Mathematics in the Modern World

Mathematical Language and Symbols

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Mathematical Language and Symbols

Like any language, mathematics has its own symbols, syntax, and rules

Learning outcomes

- Discuss the language, symbols, and conventions of mathematics.
- Explain the nature of mathematics as a language.
- Perform operations on mathematical expressions correctly.
- Acknowledge that mathematics is a useful language.

Learning the Language of Mathematics

(Jamison, 2000)

Unlike the language of ordinary speech, mathematical language is

- nontemporal
- devoid of emotional content
- precise

Example: The word "any" in ordinary speech is ambiguous.

Can anyone work this problem?	(existential qualifier)
Anyone can do it!	(universal qualifier)

An example of formal mathematical writing

Definition

A real number ϵ is said to be *small* if $\epsilon^2 < \epsilon$.

Theorem

If $\epsilon \in \mathbb{R}$ *is small, then* $0 < \epsilon < 1$.

Proof.

By Definition 1, $\epsilon^2 < \epsilon$. Assume $\epsilon < 0$. Then dividing both sides of $\epsilon^2 < \epsilon$ by ϵ results in $\epsilon > 1$, a contradiction. Thus, either $\epsilon = 0$ or $\epsilon > 0$. Assume $\epsilon = 0$. This leads to $0^2 < 0$, a contradiction. Thus, $\epsilon > 0$. Then dividing both sides of $\epsilon^2 < \epsilon$ by ϵ results in $\epsilon < 1$. Thus, $0 < \epsilon < 1$. $\square$

Definitions

(Jamison, 2000)

A definition is a *concise* statement of the *basic* properties of an object or concept which *unambiguously identify* that object or concept.

Every concept is defined as a subclass of a more general concept called the *genus*. Each special subclass of the genus is characterized by special features called the species.

Good definition

A rectangle is a *quadrilateral* all four of whose angles are right angles.

Definitions (continuation)

(Jamison, 2000)

Poor definition (not concise)

A rectangle is a *parallelogram* in which the diagonals have the same length and all the angles are right angles. It can be inscribed in a circle and its area is given by the product of two adjacent sides.

Poor definition (not basic)

A rectangle is a *parallelogram* whose diagonals have equal lengths.

Bad definition (ambiguous)

A rectangle is a *quadrilateral* with right angles.

Unacceptable definition (no genus)

rectangle: has right angles

An example of using the incorrect genus

(Karagila, 2011)

69 This is a story that I heard from one of the postdocs from my university, which in turn heard it from one of the professor at the university (I didn't bother to verify with him as the source seems relatively reliable).

The said professor was a postdoc in some university in the USA a few decades ago, and he was teaching a basic course on group theory. One of the homework assignments had a question of the form:
"Let G_1 be the group $\ldots$, and G_2 be the group $\ldots$ Prove that G_1 and G_2 are isomorphic."

One of the papers submitted had an answer "We will show that G_1 is isomorphic..." and some nonsense, followed by "Now we'll show that G_2 is isomorphic..." and more nonsense.

share cite answered Jan 31 '11 at 18:53 community wiki Asaf Karagila

86 ▲ I gave a homework problem, "Let G_1 be the group $\ldots$, let G_2 be the group $\ldots$. Are G_1 and G_2 isomorphic?" and was astonished to get the response, "G_1 is, but G_2 isn't." Are Asaf's story and mine isomorphic? – Gerry Myerson Jan 31 '11 at 22:39

165 ▲ @Gerry: Asaf's is, but yours isn't. – Nate Eldredge Feb 1 '11 at 1:20

31 ▲ You've been a lovely audience. Nate and I will be here all week. – Gerry Myerson Feb 1 '11 at 11:41

22 ▲ This is the same syntax as the joke "Oh Harry, if only we were married!" "We are, Sally... Oh, did you mean to each other?" – David Speyer Feb 1 '11 at 14:30

10 Don't forget to tip your server! – Nate Eldredge Feb 3 '11 at 15:41

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Implications

(Jamison, 2000)

If a figure is a square, then it is a rectangle.	Hypothetical
A figure is a square only if it is a rectangle.	
A figure is a rectangle whenever it is a square.	
All squares are rectangles.	Categorical
For a figure to be a square, it must necessarily be a rectangle.	Necessity
A sufficient condition for a figure to be a rectangle is that it be a square.	Sufficiency
A figure cannot be a square and fail to be a rectangle.	Conjunctive
A figure is either a rectangle or it is not a square.	Disjunctive

Implications (continuation)

(Jamison, 2000)

1. An implication is not the same as a conjunction.

   "If quadrilateral $ABCD$ is a square, then it is a rectangle."
   is not the same as
   "Quadrilateral $ABCD$ is a square and a rectangle."

2. An implication is not the same as its converse.

   "If quadrilateral $ABCD$ is a square, then it is a rectangle."
   is not the same as
   "If quadrilateral $ABCD$ is a rectangle, then it is a square."

3. The relationship between premise and conclusion is not one of causality, and the premise and conclusion can be implicit in a turn of phrase that is not an explicit if-then statement.

Aristotle "insisted that the subclasses (species) of each genus be disjoint: they could not overlap and one subclass could not include another. Thus for Aristotle, a square was NOT a rectangle. [...] From the modern point of view this is inconvenient. Virtually everything one wants to prove about non-square rectangles also holds for squares, so it is a nuisance to have to state and prove two separate theorems. The modern standard is that squares are special cases of rectangles, so theorems about rectangles also apply to squares." (Jamison, 2000, p. 53)

Disjunctions

(Gowers, 2008, p. 13)

Disjunctions in ordinary speech can be ambiguous.

A: "I like my coffee with sugar."
B: "I like my coffee without sugar."
$A \vee B$: "I like my coffee with sugar or
I like my coffee without sugar."

Is $A \vee B$ the same as "I like my coffee with or without sugar"?

Question: "Would you like your coffee with or without sugar?"
Answer: "Yes please."

Implications

(Gowers, 2008, p. 13)

Implications in ordinary speech can be ambiguous.

> *At the supper table, my young daughter once said, "Put your hand up if you're a girl." One of my sons, to tease her, put his hand up on the grounds that, since she had not added, "and keep it down if you're a boy," his doing so was compatible with her command.*

G: "I am a girl."
H: "I put my hand up."

G	H	$G \Rightarrow H$
false	false	true
false	true	true
true	false	false
true	true	true

Quantifiers

(Gowers, 2008, p. 14)

Quantifiers in ordinary speech can be ambiguous.

1. Nothing is better than lifelong happiness.
2. But a cheese sandwich is better than nothing.
3. Therefore, a cheese sandwich is better than lifelong happiness.

The word “nothing” is used differently in (1) and in (2).

1. ~~(To have) nothing is better than (to have) lifelong happiness.~~
2. But (to have) a cheese sandwich is better than (to have) nothing.

Quantifiers (continuation)

(Gowers, 2008, p. 14)

Quantifiers in ordinary speech can be ambiguous.

1. Everybody likes at least one drink, namely water.
2. Everybody likes at least one drink; I myself go for red wine.

The clause "Everybody likes at least one drink" is used differently in (1) and in (2).

1. There exists a drink D such that, for every person P, P likes D.
2. For every person P there exists a drink D such that P likes D.

Quantifiers (continuation)

(Gowers, 2008, p. 14)

"The numbers a, b, m, and n are positive integers."

$$a, b, m, n \in \mathbb{Z}^+$$

"The number m is a prime number."

$$\forall a, b,\ (ab = m) \Rightarrow ((a = 1) \vee (b = 1)),\ m > 1$$

$m \in \mathbb{P}$, where $\mathbb{P}$ is the set of prime numbers

"There are infinitely many primes."

$$\forall n, \exists m,\ (m > n) \wedge (m \in \mathbb{P})$$

Negation

(Gowers, 2008, p. 15)

Let $\mathbb{Z}^+$ be the set of positive integers and A be a set of positive integers. (That is, $A \subseteq \mathbb{Z}^+$.)

P: "Every number in the set A is odd."

$\neg P$: ~~"Every number is the set A is even."~~

P: $\forall n \in A,\ n$ is odd

$\neg P$: ~~$\forall n \in A,\ \neg(n$ is odd$)$~~

$\neg(\forall n \in A,\ n$ is odd$)$

$\exists n \in A,\ n$ is even

$\neg P$: "There exists a number is the set A that is even."

Free and bound variables

(Gowers, 2008, p. 15)

$$\forall a, b,\ (ab = m) \Rightarrow ((a = 1) \vee (b = 1))$$

The variable m is called a *free variable* because it denotes a specific object.

The variables a and b are called *bound* (or *dummy*) variables because they do not denote specific objects.

"For which values of m is the statement true?"
~~"For which values of a (or of b) is the statement true?"~~

Levels of formality

(Gowers, 2008, p. 16)

P: "Every nonempty set of positive integers has a least element."

Q: "Set A is a nonempty set of positive integers."
R: "Set A has a least element."
P: "For every set of positive integers, if it is nonempty, then it has a least element."

Q: $\exists n \in \mathbb{Z}^+,\ n \in A$
R: $\exists x \in A,\ \forall y \in A,\ (y > x) \vee (y = x)$

P: $\forall A \subseteq \mathbb{Z}^+,$
$(\exists n \in \mathbb{Z}^+,\ n \in A) \Rightarrow (\exists x \in A,\ \forall y \in A,\ (y > x) \vee (y = x))$

Sample exam questions

1. For an implication $A \Rightarrow B$, its *contrapositive* is $\neg B \Rightarrow \neg A$, its *inverse* is $\neg A \Rightarrow \neg B$, and its *converse* is $B \Rightarrow A$. Let an implication be "Every repeating decimal is a rational number." Express *in words* its
   (a) contrapositive:
   (b) inverse:
   (c) converse:

2. The statement "Even numbers are either positive or negative" is found in the Simple English Wikipedia page for "Even number." (`https://simple.wikipedia.org/w/index.php?title=Even_number&oldid=5654255`)
   (a) Express this statement *symbolically.*
   (b) Explain why the statement is incorrect.

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