Assessing Proportional Reasoning Skills and Understanding Using the Water Rectangle Task

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Introduction

Noche (2013) studied if supplemental self-paced instruction that focuses on the mastery of either concepts or procedures through repetition with variation helps young adults improve their performance in tasks designed to assess their proportional reasoning understanding and skills. He created eleven worksheets for the study, each having a conceptual and a procedural version. Noche and Vistro-Yu (2015) describe the worksheets in detail.

One worksheet involves what they call a *water rectangle*—a sealed transparent container that can be tilted from one horizontal resting position to another. The *water rectangle task* is to determine which of two identical water rectangles (in different orientations) contains more liquid.

It was inspired by the *water triangle task* which uses Kurtz's (1976, p. 34) *water triangle* to demonstrate a constant product relationship. A water triangle is "a container of colored liquid inside a right triangle where the triangle can be tilted and the water levels on the left and right side can be measured on a built-in scale" (Proportional reasoning, 2012). The water triangle task is to predict what the water level on one side will be given the water level on the other side.

Proportional Reasoning

Using Harel, Behr, Post, and Lesh's (1992) task variables, we say that the water rectangle tasks are relational propositions on extensive quantities involving the physical principle of uniform pressure of a liquid at rest, the semantic relation of partitive division, and the mathematical principle of ratio composition (a multiplicative order determination principle). The tasks in the conceptual version are nonnumeric and involve invariance of ratio (see Figure 3); those in the procedural version are numeric and involve invariance of product (see Figure 4).

All the ratios being compared are positive and less than or equal to one. In the conceptual version, the denominators are not explicit. In the procedural version, the denominators are the same. The conceptual version is presented as what Boyer, Levine, and Huttenlocher (2008) call a continuous quantity and the procedural version is presented as what Jeong, Levine and Huttenlocher (2007) call a discrete adjacent quantity.

Concepts and Procedures

The conceptual version involves number-free tasks done without arithmetic computations. Concepts are discussed using examples explained through words and pictures (see Figure 1). To discourage the numerical representation of quantities, the containers do not have any grids or tick marks.

The procedural version involves tasks done using arithmetic computations. Procedures are discussed using examples explained through numbers and pictures (see Figure 2). To encourage the use of the numeric procedure, each container has a numerical representation (a fraction) of its fullness and markings that help illustrate this fraction. (The procedure is numeric as it requires the multiplication of a shaded region's number of rows with its number of columns.)

Mastery through Repetition with Variation

The worksheets in Noche's (2013) study focus on mastery through repetition with variation, in a style similar to that of Kumon Math (Ukai, 1994).

Each worksheet is a booklet eight half-letter sized pages long to be answered individually without using books or calculators. One worksheet is to be done each day, taking around 15 to 30 minutes to complete. Students answer the worksheets at their own pace, prioritizing performance over speed. The tasks are arranged in ascending order of difficulty, with each task slightly varying from the previous one, and are to be done in the order they are presented. Students may approach the teacher for short clarifications regarding the worksheets.

At the start of each daily session, each student individually consults with the teacher who shows him or her how he or she performed in the previous worksheet. If there are few errors (10% or less), then the student corrects the errors with the help of the teacher, then answers the next worksheet alone. If there are many errors (more than 10%), then the teacher provides some brief feedback and the student repeats the whole worksheet alone. If the student takes much longer to finish a worksheet than the time allotted for it (twice the time or more), then he or she repeats the worksheet. The time allotted for each worksheet is based on the average time taken by students in a pilot study.

If the previous worksheet was done at home, then the student submits it at the start of the session and is given the next worksheet even though the submitted worksheet has not yet been checked. The submitted worksheets are checked by the teacher before the next session to determine what worksheets to assign during the next session.

Results

Noche's (2013) study used these worksheets to find empirical evidence on the causal relationships between conceptual and procedural knowledge in mathematics—how "[p]ossession of one type of knowledge is causally related to acquisition of the other" (Rittle-Johnson & Siegler, 1998, p. 78). The experimental study used a randomized pretest-post-test control group design with three groups (conceptual, procedural, and control).

The procedural group's changes in amount of procedural knowledge were significantly different from (higher than) those of the other groups (Kruskal-Wallis one-way analysis of variance by ranks $H = 12.54$, $p = .002$). But the conceptual group's changes in amount of conceptual knowledge did not significantly differ from those of the other groups ($H = 0.06$, $p = .968$). (See Figure 5.)

The procedural group spent significantly more time than the conceptual group in doing the supplemental instruction ($H = 13.69$, $p = .000$). So the results could be due to the difference in the time spent doing worksheets and not due to the difference in the type of supplemental instruction.

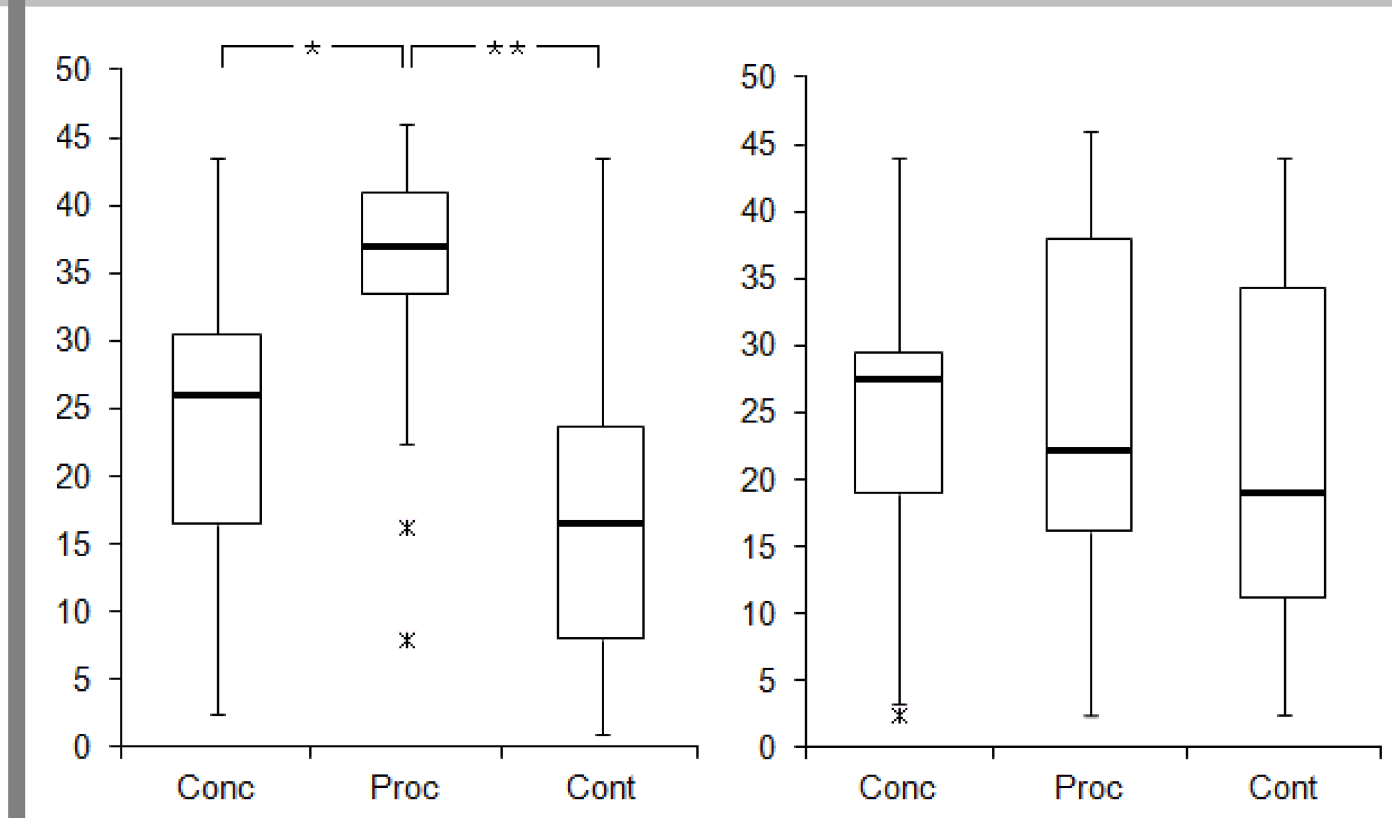


Figure 5: Changes in amount of (a) procedural knowledge and (b) conceptual knowledge by groups (*$p < .05$, **$p < .01$). Ranks of the raw data are shown.

Recommendations

The worksheets need to be further revised and tested to finally determine whether or not there is a significant difference in completion times between the two versions.

More significant improvements in understanding and skills may be obtained with a larger number of worksheets. Additional worksheets are now being planned.

Portions of the Worksheets

Ha2

Let there be a pair of identical rectangular containers. Colored liquid is put into each container, then the containers are sealed. The amount of liquid put into the containers may differ or may be the same.

Shown below is a pair of identical containers. On the left is one container lying on one side; on the right is the other container lying on its other side. The container on the left has more liquid than the one on the right.

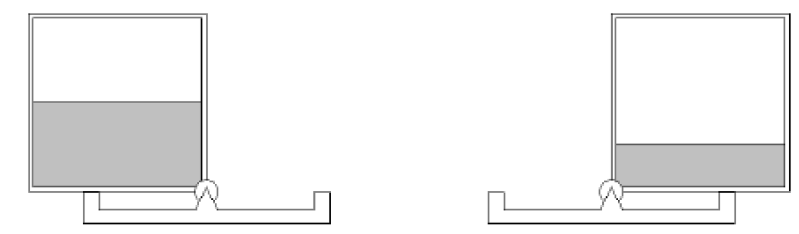


Figure 1: A conceptual discussion and example

Hb2

Let there be a pair of identical rectangular containers. Colored liquid is put into each container, then the containers are sealed. The amount of liquid put into the containers may differ or may be the same.

Shown below is a pair of identical containers. On the left is one container lying on one side; on the right is the other container lying on its other side. The container on the left has more liquid than the one on the right because the liquid on the left covers a larger fraction of the same grid (4/6) than that on the right (3/6).

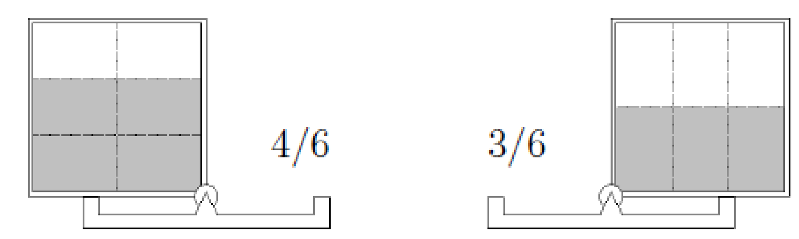
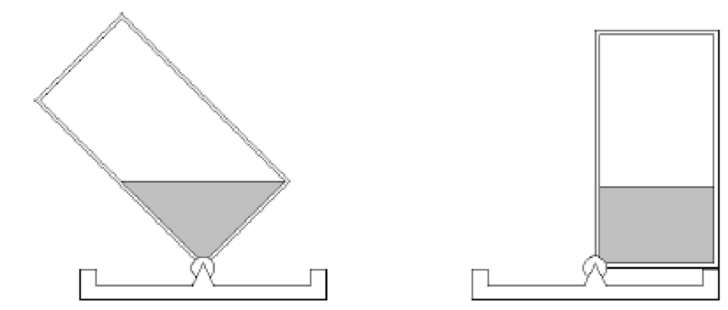


Figure 2: A procedural discussion and example

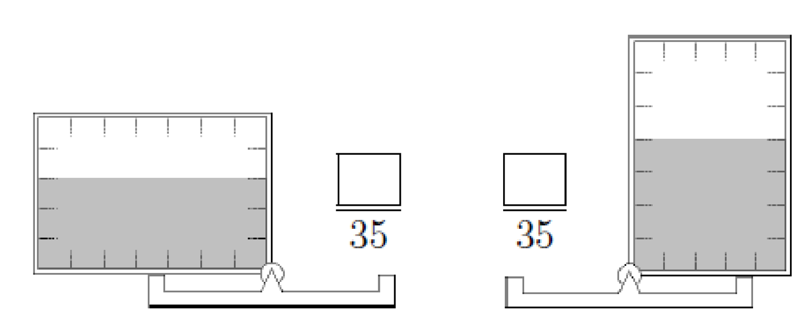
3. Which of the two containers shown below has more liquid?



(a) The container on the left has more liquid.
(b) The container on the right has more liquid.
(c) Both containers have the same amount of liquid.

Figure 3: A nonnumeric worksheet task

3. Indicate the fraction of the grid covered by the liquid in each container shown below. Which container has more liquid?



(a) The container on the left has more liquid.
(b) The container on the right has more liquid.
(c) Neither. Both have the same amount of liquid.

Figure 4: A numeric worksheet task

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