MATIMYÁS MATEMATIKA
ISSN 0115-6926

Journal of the Mathematical Society of the Philippines
Vol. 31 Nos. 1-3 (2008) pp. 25-28

Generalized Choral Sequences

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Abstract

We consider infinite binary sequences $\{c(k)\}_0^\infty$ defined by $c(3n+r_0) = 0$, $c(3n+r_1) = 1$, and $c(3n+r_c) = c(n)$ (where the r's are distinct elements of $\{0, 1, 2\}$) for all non-negative integers n, and present a characteristic function for them. These sequences are cube-free and any finite subsequence of one is either a subsequence of another or the complement of a subsequence of another.

Keywords: cube-free, choral sequence

1 Introduction

In 1995, Ian Stewart [4] presented the sequence

$$\{s(k)\}_0^\infty = 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ \ldots,$$

which he calls the choral sequence, as an example of a cube-free (binary) sequence, one that does not contain any subsequences of the form xxx, where x is a sequence of 0's and 1's. This is sequence A116178 in Sloane's Online Encyclopedia of Integer Sequences [3].

We define Stewart's choral sequence $\{s(k)\}_0^\infty$ by $s(3n) = 0$, $s(3n+2) = 1$, and $s(3n+1) = s(n)$ for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. (The presentation here differs slightly from Stewart's. The sequence he presented starts at $s(1)$ and not $s(0)$, and he used $s(3n-1) = 1$ instead of $s(3n+2) = 1$.)

Definition 1. *A* generalized choral sequence *is an infinite binary sequence* $\{c(k)\}_0^\infty$ *defined by* $c(3n+r_0) = 0$, $c(3n+r_1) = 1$, *and* $c(3n+r_c) = c(n)$ *where the r's are distinct elements of* $\{0, 1, 2\}$ *and* $n \in \mathbb{N}$.

Note that if $r_c = 0$, then the sequence is not uniquely defined, that is, $c(0)$ can be either 0 or 1.

The sequences of numbers we consider may also be thought of as words of letters.

Theorem 1. *A generalized choral sequence* $\{c(k)\}_0^\infty$ *has a characteristic function*

$$c(k) = \begin{cases} 1, & \text{if } \exists m, n \in \mathbb{N} \text{ such that } k = 3^m(3n+r_1) + \frac{r_c}{2}(3^m - 1); \\ 0, & \text{if } \exists m, n \in \mathbb{N} \text{ such that } k = 3^m(3n+r_0) + \frac{r_c}{2}(3^m - 1). \end{cases}$$

Proof. Setting $m = 0$ in $c\left(3^m\left(3n + r_0\right) + \frac{r_c}{2}\left(3^m - 1\right)\right) = 0$ yields $c(3n + r_0) = 0$ for all $n \in \mathbb{N}$. Setting $m = 0$ in $c\left(3^m\left(3n + r_1\right) + \frac{r_c}{2}\left(3^m - 1\right)\right) = 1$ yields $c(3n + r_1) = 1$ for all $n \in \mathbb{N}$.

If $c(k) = 1$, then $k = 3^m\left(3n + r_1\right) + \frac{r_c}{2}\left(3^m - 1\right)$ for some $m, n \in \mathbb{N}$ and $3k + r_c = 3^{m+1}\left(3n + r_1\right) + \frac{r_c}{2}\left(3^{m+1} - 1\right)$. Thus, if $c(k) = 1$, then $c(3k + r_c) = 1$. Similarly, it can be shown that if $c(k) = 0$, then $c(3k + r_c) = 0$. Thus, $c(3n + r_c) = c(n)$ for all $n \in \mathbb{N}$. □

Remark 1. $c(k)$ is well defined for any $k \in \mathbb{N}$. Assume otherwise, that is, assume there exist $m_a, m_b, n_a, n_b \in \mathbb{N}$ such that $3^{m_a}\left(3n_a + r_1\right) + \frac{r_c}{2}\left(3^{m_a} - 1\right) = 3^{m_b}\left(3n_b + r_0\right) + \frac{r_c}{2}\left(3^{m_b} - 1\right)$.

If $m_a = m_b$ then $3n_a + r_1 = 3n_b + r_0$. But $3n_a + r_1 \equiv r_1 \pmod 3$ and $3n_b + r_0 \equiv r_0 \pmod 3$ for any $n_a, n_b \in \mathbb{N}$. Since $r_1 \not\equiv r_0 \pmod 3$ then $3n_a + r_1 \neq 3n_b + r_0$ for any $n_a, n_b \in \mathbb{N}$. Thus, $m_a \neq m_b$.

If $m_b > m_a$, then $\left(3n_a + r_1 + \frac{r_c}{2}\right) = 3^{m_b - m_a}\left(3n_b + r_0 + \frac{r_c}{2}\right)$ and $(6n_a + 2r_1 + r_c) = 3^{m_b - m_a}(6n_b + 2r_0 + r_c)$. The right-hand side of the latter equation is a multiple of 3 but the left-hand side is not a multiple of 3 since r_1 and r_c are distinct elements of $\{0, 1, 2\}$. (If $m_a > m_b$, a similar argument yields the same result.) This contradiction means our initial assumption is wrong.

Example 1. The fixed point of the morphism specified by $0 \mapsto 010$ and $1 \mapsto 011$ iterated on 0 is found in a tutorial by Berstel and Karhumäki [1] and is sequence A080846 in Sloane's OEIS [3]. It is a generalized choral sequence $\{z(k)\}_0^\infty$ with $r_0 = 0$, $r_1 = 1$, and $r_c = 2$. Its characteristic function is

$$z(k) = \begin{cases} 1, & \text{if } \exists m, n \in \mathbb{N} \text{ such that } k = 3^m\left(3n + 1\right) + \left(3^m - 1\right); \\ 0, & \text{if } \exists m, n \in \mathbb{N} \text{ such that } k = 3^m\left(3n\right) + \left(3^m - 1\right). \end{cases}$$

Example 2. Stewart's choral sequence is the fixed point of the morphism specified by $0 \mapsto 001$ and $1 \mapsto 011$ iterated on 0. (See, for example, [2].) It is a generalized choral sequence $\{s(k)\}_0^\infty$ with $r_0 = 0$, $r_1 = 2$, and $r_c = 1$. Its characteristic function is

$$s(k) = \begin{cases} 1, & \text{if } \exists m, n \in \mathbb{N} \text{ such that } k = 3^m\left(3n + 2\right) + \frac{1}{2}\left(3^m - 1\right); \\ 0, & \text{if } \exists m, n \in \mathbb{N} \text{ such that } k = 3^m\left(3n\right) + \frac{1}{2}\left(3^m - 1\right). \end{cases}$$

We mention in passing the following theorems which generalize Theorem 1.

Theorem 2. *Let the infinite sequence* $\{a(k)\}_0^\infty$ *be defined by* $a(\ell n + r_0) = 0$, $a(\ell n + r_a) = a(n)$, *and* $a(\ell n + r_{1,1}) = a(\ell n + r_{1,2}) = \cdots = a(\ell n + r_{1,\ell-2}) = 1$ *for all* $n \in \mathbb{N}$ *(where the r's are distinct elements of* $\{0, 1, \ldots, \ell - 1\}$*). The sequence has a characteristic function*

$$a(k) = \begin{cases} 0, & \text{if } \exists m, n \in \mathbb{N} \text{ such that } k = \ell^m\left(\ell n + r_0\right) + \frac{r_a}{\ell - 1}\left(\ell^m - 1\right); \\ 1, & \text{otherwise.} \end{cases}$$

Theorem 3. *Let the infinite sequence* $\{b(k)\}_0^\infty$ *be defined by* $b(\ell n + r_1) = 1$, $b(\ell n + r_b) = b(n)$, *and* $b(\ell n + r_{0,1}) = b(\ell n + r_{0,2}) = \cdots = b(\ell n + r_{0,\ell-2}) = 0$ *for all* $n \in \mathbb{N}$ *(where the r's are distinct elements of* $\{0, 1, \ldots, \ell - 1\}$*). The sequence has a characteristic function*

$$b(k) = \begin{cases} 1, & \text{if } \exists m, n \in \mathbb{N} \text{ such that } k = \ell^m\left(\ell n + r_1\right) + \frac{r_b}{\ell - 1}\left(\ell^m - 1\right); \\ 0, & \text{otherwise.} \end{cases}$$

The proofs are similar to that of Theorem 1.

2 Some Properties

Lemma 1. *A generalized choral sequence* $\{c(k)\}_0^\infty$ *has all the subsequences* 001*,* 010*,* 011*,* 100*,* 101*, and* 110*.*

Proof. There are two cases: either $r_c + 1 \equiv r_0 \pmod 3$ and $r_c + 2 \equiv r_1 \pmod 3$, or $r_c + 1 \equiv r_1 \pmod 3$ and $r_c + 2 \equiv r_0 \pmod 3$.

In the first case, there exists a subsequence $c(3n+r_c+1)c(3n+r_c+2)c(3n+r_c+3)c(3n+r_c+4)c(3n+r_c+5) = 0\,1\,c(n+1)\,0\,1$ for some $n \in \mathbb{N}$. That is, there exist subsequences 01001 and 01101. These two subsequences contain all the subsequences 001, 010, 011, 100, 101, and 110.

In the second case, there exists a subsequence $c(3n+r_c+1)c(3n+r_c+2)c(3n+r_c+3)c(3n+r_c+4)c(3n+r_c+5) = 1\,0\,c(n+1)\,1\,0$ for some $n \in \mathbb{N}$. That is, there exist subsequences 10010 and 10110. These two subsequences contain all the subsequences 001, 010, 011, 100, 101, and 110. □

Theorem 4. *A generalized choral sequence* $v = \{c(k)\}_0^\infty$ *is cube-free.*

Proof. The proof here is practically the same as Stewart's [4] with some ideas taken from Berstel and Karhumäki [1].

Assume there is a cube xxx in v. Denote the length of the sequence x by $|x|$.

Any three consecutive terms of v must have the terms $c(k_0) = 0$ and $c(k_1) = 1$ where $k_0 \equiv r_0 \pmod 3$ and $k_1 \equiv r_1 \pmod 3$. Thus, 000 and 111 are not subsequences of v and $|x| \neq 1$.

From the discussion in the proof of Lemma 1, any nine consecutive terms of v must have a subsequence either of the form $c(k)\,0\,1\,c(k+1)\,0\,1\,c(k+2)$ or of the form $c(k)\,1\,0\,c(k+1)\,1\,0\,c(k+2)$. Since $c(k)$, $c(k+1)$, and $c(k+2)$ are not all the same for any $k \in \mathbb{N}$, then $|x| \neq 3$. We may restate this result as $|x| = 3$ if and only if there exists a cube yyy with $|y| = 1$. Since there is no cube yyy with $|y| = 1$, then $|x| \neq 3$.

Any $9p$ consecutive terms of v, where p is a positive integer, must have a subsequence either of the form $c(k)\,0\,1\,c(k+1)\ldots0\,1\,c(k+3p-1)$ or of the form $c(k)\,1\,0\,c(k+1)\ldots1\,0\,c(k+3p-1)$. Thus, $|x| = 3p$ if and only if there exists a cube yyy with $|y| = p$.

The word x starts at $c(k)$, $c(k+|x|)$, and $c(k+2|x|)$ for some $k \in \mathbb{N}$. If $|x|$ is not a multiple of 3, then k, $k+|x|$, and $k+2|x|$ take all the possible values modulo 3 and $c(k)$, $c(k+|x|)$, and $c(k+2|x|)$ cannot all be the same. Thus, $|x|$ is a multiple of 3.

If $|x|$ is a positive multiple of 3, then it can be expressed as $3^a \cdot b$, where a is a positive integer and b is a positive integer that is not a multiple of 3. Since $|x| = 3p$ if and only if there exists a cube yyy with $|y| = p$, we may use this repeatedly to get the result that there exists a cube yyy with $|y| = b$. This contradicts our earlier result that if there exists a cube yyy, then $|y|$ is a multiple of 3. Therefore, $|x|$ is not a positive multiple of 3.

Thus, $|x| = 0$ and v is cube-free. □

Theorem 5. *Let* $v = \{c_v(k)\}_0^\infty$ *and* $w = \{c_w(k)\}_0^\infty$ *be generalized choral sequences such that* $r_c + 1 \equiv r_0 \pmod 3$ *and* $r_c + 2 \equiv r_1 \pmod 3$ *for both. Any finite subsequence of* v *is also a subsequence of* w*.*

Proof. By Theorem 4, 000 and 111 are not subsequences of v or w. By Lemma 1, all the other three-term binary sequences are subsequences of v and w. Thus, any three-term subsequence of v is also a subsequence of w. That is, for a given k_v, there exists a k_w such that $c_v(k_v)c_v(k_v+1)c_v(k_v+2) = c_w(k_w)c_w(k_w+1)c_w(k_w+2)$.

For a given k_v, the sequence $c_v(k_v)\,0\,1\,c_v(k_v+1)\,0\,1\,c_v(k_v+2)$ is a subsequence of v. Similarly, for some k_w, the sequence $c_w(k_w)\,0\,1\,c_w(k_w+1)\,0\,1\,c_w(k_w+2)$ is a subsequence of w. By the previous discussion, there exists a k_w such that $c_w(k_w)\,0\,1\,c_w(k_w+1)\,0\,1\,c_w(k_w+2)$ is a subsequence of w which is the same as $c_v(k_v)\,0\,1\,c_v(k_v+1)\,0\,1\,c_v(k_v+2)$.

Consequently, for a given k_v there exists a k_w such that $c_w(k_w)\,0\,1\,c_w(k_w+1)\cdots 0\,1\,c_w(k_w+6)$ is a subsequence of w which is the same as $c_v(k_v)\,0\,1\,c_v(k_v+1)\cdots 0\,1\,c_v(k_v+6)$, a subsequence of v.

We can extend this reasoning to arbitrarily long finite sequences of similar form. Any finite subsequence of v is a subsequence of a sequence of this form. Thus, any finite subsequence of v is also a subsequence of w. $\square$

Theorem 6. *Let v and w be generalized choral sequences such that $r_c+1 \equiv r_1 \pmod 3$ and $r_c+2 \equiv r_0 \pmod 3$ for both. Any finite subsequence of v is also a subsequence of w.*

The proof of Theorem 6 is similar to that of Theorem 5, but now sequences of the form $c(k)\,1\,0\,c(k+1)\,1\,0\,c(k+2)$ are considered.

Definition 2. *We define the* complement *of a binary sequence x to be the sequence $\overline{x}$ obtained by replacing each 0 in x with a 1, and each 1 in x with a 0.*

Corollary 1. *Let v and w be generalized choral sequences. Any finite subsequence of v is either a subsequence of w or the complement of a subsequence of w.*

Proof. If v and w satisfy the conditions of Theorem 5 or of Theorem 6, then any finite subsequence of v is a subsequence of w.

Otherwise, one of them, say v, has $r_c+1 \equiv r_0 \pmod 3$ and $r_c+2 \equiv r_1 \pmod 3$ and the other one, say w, has $r_c+1 \equiv r_1 \pmod 3$ and $r_c+2 \equiv r_0 \pmod 3$. By Lemma 1, any three-term subsequence of v is the complement of a subsequence of w. That is, for a given k_v, there exists a k_w such that $c_v(k_v)c_v(k_v+1)c_v(k_v+2) = \overline{c}_w(k_w)\overline{c}_w(k_w+1)\overline{c}_w(k_w+2)$.

Furthermore, there exists a subsequence of w $c_w(k_w)\,1\,0\,c_w(k_w+1)\,1\,0\,c_w(k_w+2)$ whose complement $\overline{c}_w(k_w)\,\overline{1}\,\overline{0}\,\overline{c}_w(k_w+1)\,\overline{1}\,\overline{0}\,\overline{c}_w(k_w+2)$ is the same as $c_v(k_v)\,0\,1\,c_v(k_v+1)\,0\,1\,c_v(k_v+2)$, a subsequence of v.

Extending this reasoning to arbitrarily long finite sequences of similar form yields the result that any finite subsequence of v is the complement of a subsequence of w. $\square$

Acknowledgment. I thank an anonymous colleague for suggesting the generalizations in Theorems 2 and 3.

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